ROTATION TOPOLOGICAL FACTORS OF MINIMAL $\mathbb{Z}^d$-ACTIONS ON THE CANTOR SET

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Abstract. In this paper we study conditions under which a free minimal $\mathbb{Z}^d$-action on the Cantor set is a topological extension of the action of d rotations, either on the product $\mathbb{T}^d$ of d 1-tori or on a single 1-torus $\mathbb{T}^1$. We extend the notion of *linearly recurrent* systems defined for $\mathbb{Z}$-actions on the Cantor set to $\mathbb{Z}^d$-actions and we derive in this more general setting, a necessary and sufficient condition, which involves a natural combinatorial data associated with the action, allowing the existence of a rotation topological factor of one these two types.

1. Introduction

Let $(X, \mathcal{A})$ be a $\mathbb{Z}^d$-action (by homeomorphisms) on a compact metric space X. The action is *free* if $\mathcal{A}(\bar{n}, x) = x$ for some $\bar{n} \in \mathbb{Z}^d$ and $x \in X$ implies $\bar{n} = 0$ and is *minimal* if the orbit of any point $x \in X$, $O_{\mathcal{A}}(x) = \{\mathcal{A}(\bar{n}, x) : \bar{n} \in \mathbb{Z}^d\}$, is dense in X.

The simplest non trivial examples of free minimal $\mathbb{Z}^d$-actions on a compact metric space are given by "rotation-type" actions on compact topological groups. This type of factors play a central role in topological dynamics of $\mathbb{Z}^d$-actions since in particular they determine weak mixing property through the existence of continuous eigenvalues. In this paper, we focus on two kinds of "rotation-type" factors that we describe now.

- First consider the $\mathbb{Z}^d$-action generated by d rotations on the product d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$, each rotation acting on $\mathbb{T}^1$. More precisely, take $\bar{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ and let $\mathcal{A}^d_{\bar{\theta}} : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}^d$ be the map defined by:
$$\mathcal{A}^d_{\bar{\theta}}(\bar{n}, x) = x + [\bar{n}, \bar{\theta}] \mod \mathbb{Z}^d \ ,$$
for $\bar{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $x \in \mathbb{T}^d$ and where $[\bar{n}, \bar{\theta}] = (n_1 \cdot \theta_1, \ldots, n_d \cdot \theta_d)$. This construction yields a minimal $\mathbb{Z}^d$-action $(\mathbb{O}^d, \mathcal{A}^d_{\bar{\theta}})$ on the closure $\mathbb{O}^d$ of the orbit of 0 in the d-torus $\mathbb{T}^d$. When the coordinates of $\bar{\theta}$ are rationally independent, the set $\mathbb{O}^d$ is the d-torus $\mathbb{T}^d$ and the action is free.
- The same $\bar{\theta}$ can be used to define a $\mathbb{Z}^d$-action on $\mathbb{T}^1$. Consider the map $\mathcal{A}^1_{\bar{\theta}} : \mathbb{Z}^d \times \mathbb{T}^1 \to \mathbb{T}^1$ given by
$$\mathcal{A}^1_{\bar{\theta}}(\bar{n}, t) = t + < \bar{n}, \bar{\theta} > \mod \mathbb{Z},$$
where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^d$. The $\mathbb{Z}^d$-action $(\mathbb{O}^1, \mathcal{A}^1_{\bar{\theta}})$ on the closure $\mathbb{O}^1$ of the orbit of 0 in the 1-torus $\mathbb{T}^1$ is again minimal. When

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the coordinates of $\bar{\theta}$ are independent on $\mathbb{Q}$, the set $\mathbb{O}^1$ is the 1-torus $\mathbb{T}^1$ and the action is free.

Assume X is a Cantor set, *i.e.*, it has a countable basis of closed and open (clopen) sets and has no isolated points (or equivalently, it is a totally disconnected compact metric space with no isolated points).

The main question we address in this paper is to determine whether a free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X is an extension of an action of type $(\mathbb{O}^d, \mathcal{A}^d_{\bar{\theta}})$ or $(\mathbb{O}^1, \mathcal{A}^1_{\bar{\theta}})$ for some $\bar{\theta} \in \mathbb{R}^d$.

Notice that a complete combinatorial answer to this question is given in [BDM] in the particular case when the dimension $d = 1$ and when the free minimal $\mathbb{Z}$-action is *linearly recurrent*. The linear recurrence of a given $\mathbb{Z}$-action is a property that involves the combinatorics of *return times* associated with a nested sequence of clopen sets (for further references on linearly recurrent $\mathbb{Z}$-actions see [CDHM],[Du1] and [Du2]).
The notion of return time to a clopen set can be generalized to $\mathbb{Z}^d$-actions when $d \geq 2$. In this case, the combinatorics of the return times associated with a nested sequence of clopen sets inherits a richer structure than in the case $d = 1$. However, as for $d = 1$, there exists a natural definition of linearly recurrent $\mathbb{Z}^d$-action. These generalizations are developed in Section 2 which is devoted to the combinatorics of return times (for further references on the structure of return times associated with a $\mathbb{Z}^d$-action see [BG] where the hierarchical ideas used in this paper are introduced, see also [S] and [SW] for related topics).
This combinatorial approach allows us to derive a necessary condition on the action to be an extension of an action of one of the two rotations described above. In the case of a linearly recurrent action this condition is sufficient. This result is given in Section 3 (Theorem 3.1) together with its proof.

2. Combinatorics of return times

Let us start this section with some general considerations.
Let $\mathbb{R}^d$ be the Euclidean d-space and $\|-\|$ its Euclidean norm. Consider two positive numbers r and R. An (r, R)-*Delone set* is a subset $\mathcal{D}$ of the d-space $\mathbb{R}^d$ equipped with the Euclidean norm $\| - \|$, which satisfies the following two properties:

(i) *Uniformly Discrete:* each open ball with radius r in $\mathbb{R}^d$ contains at most one point in $\mathcal{D}$;
(ii) *Relatively Dense:* each open ball with radius R contains at least one point in $\mathcal{D}$.

When the constants r and R are not explicitly used, we will say in short *Delone set* for an (r, R)-Delone set. We refer to [LP] for a more detailed approach of the theory of Delone sets.
A *patch* of a Delone set $\mathcal{D}$ is a finite subset of $\mathcal{D}$. A Delone set is of *finite type* if for each $M > 0$, there exist only finitely many patches in $\mathcal{D}$ of diameter smaller than M up to translation. Finally, a Delone set of finite type is *repetitive* if for each patch P in $\mathcal{D}$, there exists $M > 0$ such that each ball with radius M in $\mathbb{R}^d$ contains a translated copy of P in $\mathcal{D}$.
Let x be a point of a Delone set $\mathcal{D}$. The *Voronoi cell* $\mathcal{V}_x$ associated with x is the convex closed set in $\mathbb{R}^d$ defined by:

$$\mathcal{V}_x = \{y \in \mathbb{R}^d \ : \ \forall\, x' \in \mathcal{D},\ \|y - x\| \leq \|y - x'\| \ \} \ .$$

The union $\cup_{x\in\mathcal{D}}\mathcal{V}_x$ is a cover of $\mathbb{R}^d$. We say that two points x and x' in $\mathcal{D}$ are *neighbors* if $\mathcal{V}_x \cap \mathcal{V}_{x'} \neq \emptyset$.
The set of return vectors associated with $\mathcal{D}$ is defined by:

$$\vec{\mathcal{D}} = \{x - y \ : \ (x, y) \in \mathcal{D} \times \mathcal{D}\} \ .$$

Lemma 2.1. *Let $\mathcal{D}$ be a Delone set of finite type. Then, there exists a finite collection $\vec{\mathcal{F}}$ of vectors in $\vec{\mathcal{D}}$ such that:*

- *$\vec{\mathcal{F}} = -\vec{\mathcal{F}}$;*
- *any vector in $\vec{\mathcal{D}}$ is a linear combination with non negative integer coefficients of vectors in $\vec{\mathcal{F}}$.*

Proof. When $\mathcal{D}$ is a Delone set of finite type, the set of vectors

$$\vec{\mathcal{F}} = \bigcup_{(x,x')\in\mathcal{D}\times\mathcal{D},(x,x') \text{ neighbors}} (x - x')$$

is finite, satisfies $\vec{\mathcal{F}} = -\vec{\mathcal{F}}$ and clearly any vector in $\vec{\mathcal{D}}$ is a linear combination with non negative integer coefficients of vectors in $\vec{\mathcal{F}}$. $\square$

Given such a set $\vec{\mathcal{F}}$, we can define the $\vec{\mathcal{F}}$-*distance* $d_{\vec{\mathcal{F}}}(x, x')$ as the minimal number of vectors in $\vec{\mathcal{F}}$ (counted with multiplicity) needed to write $x - x'$ for $x, x' \in \mathcal{D}$. The $\vec{\mathcal{F}}$-*diameter* of a patch P, denoted by $diam_{\vec{\mathcal{F}}}(P)$, is the maximal $\vec{\mathcal{F}}$-distance of pair of points in $\mathcal{D}$.

Consider now a free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X. Let $\mathcal{C}$ be a clopen set in X and y a point in $\mathcal{C}$. The set of *return times* of the orbit of y in $\mathcal{C}$ is defined by

$$\mathcal{R}_\mathcal{C}(y) = \{\bar{n} \in \mathbb{Z}^d \ : \ \mathcal{A}(\bar{n}, y) \in \mathcal{C}\} \ .$$

Proposition 2.2. *The set of return times $\mathcal{R}_\mathcal{C}(y)$ is a repetitive Delone set of finite type in $\mathbb{Z}^d$. Furthermore, if y and y' are two points in $\mathcal{C}$, the sets $\mathcal{R}_\mathcal{C}(y)$ and $\mathcal{R}_\mathcal{C}(y')$ have the same patches up to translation.*

Proof. • *$\mathcal{R}_\mathcal{C}(y)$ is a Delone set of finite type.*
The minimality of the action implies that the orbit of any point in X visits $\mathcal{C}$. For each $x \in X$ consider $\bar{n}_x \in \mathbb{Z}^d$ be such that $\mathcal{A}(\bar{n}_x, x)$ is in $\mathcal{C}$. Since $\mathcal{C}$ is open, there exists a small neighborhood U_x of x such that for any x' in U_x we also have $\mathcal{A}(\bar{n}_x, x') \in \mathcal{C}$. Therefore $\{U_x \ : \ x \in X\}$ is a cover of X. Since X is compact, we can extract a finite cover $\{U_{x_i} \ : \ i \in I\}$. Let us choose $R > \max_{i\in I} \|\bar{n}_{x_i}\|$. It is clear that any ball with radius R in $\mathbb{R}^d$ intersects $\mathcal{R}_\mathcal{C}(y)$. Thus, $\mathcal{R}_\mathcal{C}(y)$ is relatively dense. Since it is a subset of $\mathbb{Z}^d$, it is a Delone set of finite type.
• *$\mathcal{R}_\mathcal{C}(y)$ is repetitive*[1].
Consider a patch P in $\mathcal{R}_\mathcal{C}(y)$, choose $\bar{n}_0$ in P and let $z = \mathcal{A}(\bar{n}_0, y) \in \mathcal{C}$. Choose now a clopen set $\mathcal{C}_z$ containing z, small enough so that for any z' in $\mathcal{C}_z$, $\mathcal{A}(\bar{n} - \bar{n}_0, z')$ is in $\mathcal{C}$ for each $\bar{n}$ in P. The set $\mathcal{R}_{\mathcal{C}_z}(z)$ is relatively dense, let R_1 be its R-constant. Let M stand for the diameter of P and let us prove that any ball with radius $R_1 + M$

[1]The proof that minimality implies repetitivity is classical and works in a more general situation. However, for sake of completeness, we fix it here for our specific context.

in $\mathbb{R}^d$ contains a translation of the patch P. Indeed, given such a ball B, choose an element $\bar{m} \in \mathcal{R}_{\mathcal{C}_z}(z)$ in the corresponding centered sub-ball of radius R_1, then by construction $\bar{m} + P$ belongs to $\mathcal{R}_{\mathcal{C}}(y)$ and to the ball B.

• *$\mathcal{R}_{\mathcal{C}}(y)$ and $\mathcal{R}_{\mathcal{C}}(y')$ have the same patches up to translation.*

Let P be a patch of $\mathcal{R}_{\mathcal{C}}(y)$ and $\bar{n}_0$ be a point in P. The minimality of the action implies that the orbit of y' accumulates on $z = \mathcal{A}(\bar{n}_0, y)$. This means that there exists $\bar{n}_1 \in \mathbb{Z}^d$ such that $\mathcal{A}(\bar{n}_1 + \bar{n} - \bar{n}_0, y')$ is in $\mathcal{C}$ when $\bar{n}$ is in P. Thus a translation of the patch P is in $\mathcal{R}_{\mathcal{C}}(y')$. □

The set of *return vectors* associated with $\mathcal{C}$ is defined by:

$$\vec{\mathcal{R}}_{\mathcal{C}} = \mathcal{R}_{\mathcal{C}}(y) - \mathcal{R}_{\mathcal{C}}(y) = \{\bar{n} - \bar{m} \ : \ (\bar{n}, \bar{m}) \in \mathcal{R}_{\mathcal{C}}(y) \times \mathcal{R}_{\mathcal{C}}(y)\} \ .$$

The fact that for any pair of points y and y' in $\mathcal{C}$, the patches of $\mathcal{R}_{\mathcal{C}}(y)$ and $\mathcal{R}_{\mathcal{C}}(y')$ fit up to translation, implies that $\vec{\mathcal{R}}_{\mathcal{C}}$ does not depend on y in $\mathcal{C}$, as suggested by the notation. Lemma 2.1 and Proposition 2.2 yield the following corollary.

Corollary 2.3. *There exists in $\vec{\mathcal{R}}_{\mathcal{C}}$ a finite collection of vectors $\vec{\mathcal{F}}_{\mathcal{C}}$ such that:*

- $\vec{\mathcal{F}}_{\mathcal{C}} = -\vec{\mathcal{F}}_{\mathcal{C}}$;
- *any vector in $\vec{\mathcal{R}}_{\mathcal{C}}$ is a linear combination with non negative integer coefficients of vectors in $\vec{\mathcal{F}}_{\mathcal{C}}$.*

Such a set $\vec{\mathcal{F}}_{\mathcal{C}}$ is called a *set of first return vectors* associated with $\mathcal{C}$.

Now we shall construct a combinatorial data associated to a $\mathbb{Z}^d$-action. Let x be a point in X and consider a sequence of nested clopen sets $X = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_n$ such that

$$\bigcap_{n \geq 0} \mathcal{C}_n \ = \ \{x\} \ .$$

Consider also the associated sets of return times $\mathcal{R}_{\mathcal{C}_n}(x)$, of return vectors $\vec{\mathcal{R}}_{\mathcal{C}_n}$ and of first return vectors $\vec{\mathcal{F}}_{\mathcal{C}_n}$ that we denote respectively (in short) by $\mathcal{R}_n(x)$, $\vec{\mathcal{R}}_n$ and $\vec{\mathcal{F}}_n$.

Proposition 2.4. *For each $n \geq 0$, there exist a constant $k(n) > 0$ and a partition of $\mathcal{R}_n(x)$ in disjoint patches $\{\mathcal{P}_n(\bar{m})\}_{\bar{m} \in \mathcal{R}_{n+1}(x)}$ such that, for each $\bar{m} \in \mathcal{R}_{n+1}(x)$:*

(i) *$\mathcal{P}_n(\bar{m}) \cap \mathcal{R}_{n+1}(x) = \{\bar{m}\}$;*
(ii) *$diam_{\vec{\mathcal{F}}_n}(\mathcal{P}_n(\bar{m})) \leq k(n)$.*

Proof. For any point $\bar{m}$ in $\mathcal{R}_{n+1}(x)$ consider its Voronoi cell $\mathcal{V}_{\bar{m},n+1}$. The intersection of this Voronoi cell with $\mathcal{R}_n(x)$ defines a patch $\mathcal{P}_n(\bar{m})$ which intersects $\mathcal{R}_{n+1}(x)$ at $\bar{m}$. It may occasionally happen that a point $\bar{l}$ in $\mathcal{R}_n(x)$ belongs to more than one Voronoi cell $\mathcal{V}_{\bar{m},n+1}$. It this case, we make an arbitrary choice to exclude the point $\bar{l}$ from all the patches it belongs to but one. This surgery done, the collection of patches $\{\mathcal{P}_n(\bar{m})\}_{\bar{m} \in \mathcal{R}_{n+1}(x)}$ realizes a partition of $\mathcal{R}_n(x)$. Furthermore, since $\mathcal{R}_{n+1}(x)$ and $\mathcal{R}_n(x)$ are repetitive Delone sets, the Euclidean diameters of the cells $\mathcal{V}_{\bar{m},n+1}$ are bounded independently of $\bar{m}$, and thus their $\vec{\mathcal{F}}_n$-diameters are bounded independently of $\bar{m}$. □

The data $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ is called *a combinatorial data* associated with the action $(X, \mathcal{A})$.
We remark that Proposition 2.4 does not require any condition on the nested sequence of clopen sets. By forgetting some $\mathcal{C}_n$'s in the sequence, it is always possible to insure the following two extra properties for the combinatorial data:

(iii) for each $n \geq 0$ and for each $\bar m$ in $\mathcal{R}_{n+1}(x)$
$$\vec{\mathcal{F}}_n \subseteq \mathcal{P}_n(\bar m) - \mathcal{P}_n(\bar m);$$
(iv) for each $n \geq 0$ and for each $\bar m$ in $\mathcal{R}_{n+2}(x)$, all the patches $\mathcal{P}_n(\bar m)$ are identical up to translation.

In this case, we say that the combinatorial data
$$(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$$
is *well distributed*.
Let m and n be two integers such that $0 \leq n \leq m$, and let $\bar p$ be a point in $\mathcal{R}_m(x)$. We denote by $\mathcal{P}_n^m(\bar p)$ the patch in $\mathcal{R}_n(x)$ defined recursively by:
$$\mathcal{P}_{m-1}^m(\bar p) = \mathcal{P}_{m-1}(\bar p) \ ,$$
and
$$\mathcal{P}_n^m(\bar p) = \cup_{\bar q\in\mathcal{P}_{n+1}^m(\bar p)}\mathcal{P}_n(\bar q) \ .$$
We adopt the convention $\mathcal{P}_m^m(\bar p) = \{\bar p\}$. The proof of the following result is plain.

Corollary 2.5. *For any $n_0 \geq 0$ and any $\bar p$ in $\mathcal{R}_{n_0}(x)$, there exists a unique $m_0 \geq n_0$ and a unique sequence $\{\bar p_l\}_{0\leq l\leq m_0-n_0}$ of points in $\mathbb{Z}^d$ such that:*

- *m_0 is the smallest $m \geq n_0$ for which $\bar p \in \mathcal{P}_{n_0}^m(0)$;*
- *$\bar p_0 = 0$;*
- *$\bar p_l \in \mathcal{P}_{m_0-l}(\bar p_{l-1})$ and $\bar p \in \mathcal{P}_{n_0}^{m_0-l}(\bar p_l)$ for all $1 \leq l \leq m_0 - n_0$;*
- *$\bar p_{m_0-n_0} = \bar p$.*

When the constant $k(n)$ in Proposition 2.4 is bounded independently on n, we say that the free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X is *linearly recurrent*. In this case, the combinatorial data $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ is said *adapted* to the action.

3. Main results

To each vector $\bar\theta$ in $\mathbb{R}^d$ we associate the linear maps $c^1_{\bar\theta} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{T}^1)$ and $c^d_{\bar\theta} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{T}^d)$ defined by
$$c^1_{\bar\theta}(\bar p) = <\bar\theta, \bar p> \ mod\, \mathbb{Z} \ \text{ and } \ c^d_{\bar\theta}(\bar p) = [\bar\theta, \bar p]\, mod\, \mathbb{Z}^d$$
for each $\bar p$ in $\mathbb{Z}^d$.
Consider a minimal free $\mathbb{Z}^d$-action $(X, \mathcal{A})$ on the Cantor set X and a combinatorial data $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ associated with this action. For any $n \geq 0$ and any $\bar\theta \in \mathbb{R}^d$ we define the *$\bar\theta$-length of $\vec{\mathcal{F}}_n$ of dimension* 1 *and* d respectively by:
$$l^1_{n,\bar\theta} = \max_{r_n\in\vec{\mathcal{F}}_n} |||c^1_{\bar\theta}(r_n)||| \ \text{ and } \ l^d_{n,\bar\theta} = \max_{r_n\in\vec{\mathcal{F}}_n} |||c^d_{\bar\theta}(r_n)|||,$$

where $|||\cdot|||$ stands for the Euclidean distance to 0 on the k-torus, $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $k = 1, d$.. The following theorem is the main result of this paper.

Theorem 3.1. *Let $(X, \mathcal{A})$ be a free minimal $\mathbb{Z}^d$-action on the Cantor set X, $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ be an associated combinatorial data and $k = 1$ or $k = d$.*

- (i) *Assume that for some $\bar{\theta} \in \mathbb{R}^d$, $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar{\theta}})$. Assume furthermore that the combinatorial data is well distributed. Then the series $\displaystyle\sum_{n\geq 0} l^k_{n,\bar{\theta}}$ converges.*
- (ii) *Conversely assume that the action is linearly recurrent, that the combinatorial data is adapted to the action and that, for some $\bar{\theta} \in \mathbb{R}^d$, the series $\displaystyle\sum_{n\geq 0} l^k_{n,\bar{\theta}}$ converges. Then $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar{\theta}})$.*

Remark 1: In the particular case when the $\mathbb{Z}^d$-action $\mathcal{A}$ is the product of d linearly recurrent $\mathbb{Z}$-actions on X, Theorem 3.1 for $k = d$ is a direct corollary of its $d = 1$ version proved in [BDM].

Remark 2: The lie group structure of $\mathbb{T}^k$ allows us to construct a continuous surjective map $\phi : \mathbb{T}^d \to \mathbb{T}^1$ defined by $\phi(\alpha_1, \ldots, \alpha_d) = \alpha_1 + \cdots + \alpha_d$. Assume that $h : (X, \mathcal{A}) \to (\mathbb{O}^d, \mathcal{A}^d_{\bar{\theta}})$ is an extension, then the map $\phi \circ h : (X, \mathcal{A}) \to (\mathbb{O}^1, \mathcal{A}^1_{\bar{\theta}})$ is also an extension. This is coherent with the fact that the convergence of the series $\displaystyle\sum_{n\geq 0} l^d_{n,\bar{\theta}}$ implies the convergence of the series $\displaystyle\sum_{n\geq 0} l^1_{n,\bar{\theta}}$.

Proof of Theorem 3.1. The proofs of both assertions of Theorem 3.1 for $k = 1$ or $k = d$ follow the same scheme and will be gathered in a single demonstration. Let $<< \cdot, \cdot >>$ stand for $[\cdot, \cdot] \mod \mathbb{Z}^d$ when $k = d$ and for $< \cdot, \cdot > \mod \mathbb{Z}$ when $k = 1$.

(i) Assume that the free minimal $\mathbb{Z}^d$-action $(X, \mathcal{A})$ is an extension of the action $\mathcal{A}^k_{\bar{\theta}}$ on the closure $\mathbb{O}^k$ of the orbit of the point 0 in the k-torus $\mathbb{T}^k$ for some $\bar{\theta}$ in $\mathbb{R}^d$. Let us denote by $h : X \to \mathbb{O}^k$ the extension. Choose a well distributed associated combinatorial data

$$(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_{n+1}(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$$

and fix $h(x) = 0 \in \mathbb{T}^k$.

For each $n \geq 0$ let v_n be the first return vector in $\vec{\mathcal{F}}_n$ such that:

$$l^k_{n,\bar{\theta}} = \max_{u_n\in\vec{\mathcal{F}}_n} |||c^k_{\bar{\theta}}(u_n)||| = |||c^k_{\bar{\theta}}(v_n)|||.$$

The following observation is a direct consequence of the continuity of h.

Lemma 3.1. *The quantity $l^k_{n,\bar{\theta}}$ goes to 0 as n goes to ∞. Furthermore, for each $\epsilon > 0$ there exists $N > 0$ such that for any pair of points $(\bar{n}, \bar{m})$ in $\mathcal{R}_N(x) \times \mathcal{R}_N(x)$, we have:*

$$|||h(\mathcal{A}(\bar{n}, x)) - h(\mathcal{A}(\bar{m}, x))||| \leq \epsilon.$$

Let B be the open ball on the k-torus, centered at 0, with radius $\sqrt{k}/2$. Fix $0 < \epsilon < \sqrt{k}/2$ and let N verifying the conclusion of Lemma 3.1 for this ϵ and such that $l^k_{n,\bar{\theta}} \leq \epsilon$ for $N \leq n$. The ball B is decomposed in 2^k sectors $S_{\epsilon_1,\ldots,\epsilon_k}$ with $\epsilon_i \in \{-1, 1\}$ for $i \in \{1, \ldots, k\}$ defined by

$$S_{\epsilon_1,\dots,\epsilon_k} = \{(x_1,\dots,x_k) \in B \ : \ x_i \cdot \epsilon_i \geq 0 \ , \ \forall i \in \{1,\dots,k\}\} \ .$$

Let $I_{\epsilon_1,\dots,\epsilon_k}$ be the set of integers n such that $c^k_{\bar\theta}(v_n)$ is in $S_{\epsilon_1,\dots,\epsilon_k}$ and let us prove that the series $\displaystyle\sum_{n\in I_{\epsilon_1,\dots,\epsilon_k}} l^k_{n,\bar\theta}$ converges. Actually, we only need to prove that the series $\displaystyle\sum_{n\in I_{1,\dots,1}} l^k_{n,\bar\theta}$ converges, a similar proof works for the other cases. This sum can be spitted into two parts:

$$\sum_{n\in I_{1,\dots,1}} l^k_{n,\bar\theta} = \sum_{n\in I_{1,\dots,1},\,\mathrm{even}} l^k_{n,\bar\theta} + \sum_{n\in I_{1,\dots,1},\,\mathrm{odd}} l^k_{n,\bar\theta}.$$

Here again we only need to prove that the series $\displaystyle\sum_{n\in I_{1,\dots,1},\,\mathrm{even}} l^k_{n,\bar\theta}$ converges, a similar proof works also for the case where n is odd. Observe that we are assuming $I_{1,\dots,1}$ is infinite.

The proof splits in five steps:

Step 1: Fix an even integer N_0 big enough in $I_{1,\dots,1}$, and let $N < n_l < n_{l-1} < \cdots < n_1 < N_0$ be the ordered sequence of even integers bigger than N that belong to $I_{1,\dots,1}$.

Step 2: Consider two points $\bar m_1$ and $\bar p_1$ in $\mathcal{R}_{n_1}(x)$ such that

$$v_{n_1} = \bar p_1 - \bar m_1.$$

Since the combinatorial data is well distributed, the two patches $\mathcal{P}_{n_2}(\bar m_1)$ and $\mathcal{P}_{n_2}(\bar p_1)$ are identical up to translation and there exists a pair of points $(\bar m_2, \bar m_2')$ in $\mathcal{P}_{n_2}(\bar m_1) \times \mathcal{P}_{n_2}(\bar m_1)$ such that

$$v_{n_2} = \bar m_2' - \bar m_2.$$

We define $\bar p_2$ in $\mathcal{P}_{n_2}(\bar p_1)$ by $\bar p_2 - \bar p_1 = \bar m_2 - \bar m_1 + v_{n_2}$. We have:

$$\bar p_2 - \bar m_2 \ = \ v_{n_1} + v_{n_2}.$$

Step 3: Since the combinatorial data is well distributed, the two patches $\mathcal{P}_{n_3}(\bar m_2)$ and $\mathcal{P}_{n_3}(\bar p_2)$ are identical up to translation and there exists a pair of points $(\bar m_3, \bar m_3')$ in $\mathcal{P}_{n_3}(\bar m_3) \times \mathcal{P}_{n_2}(\bar m_2)$ such that

$$v_{n_3} = \bar m_3' - \bar m_3.$$

We define $\bar p_3$ in $\mathcal{P}_{n_2}(\bar p_2)$ by $\bar p_3 - \bar p_2 = \bar m_3 - \bar m_2 + v_{n_3}$. We have:

$$\bar p_3 - \bar m_3 \ = \ v_{n_1} + v_{n_2} + v_{n_3}.$$

Step 4: We iterate this construction until we get the points $\bar m_l$ and $\bar p_l$ which satisfy:

$$\bar p_l - \bar m_l \ = \ \sum_{j=1}^{l} v_{n_j}.$$

Step 5: We have:

$$|||h(\mathcal{A}(\bar{p}_l, x)) \; - \; h(\mathcal{A}(\bar{m}_l, x))||| \; = ||| << \sum_{j=1}^{l} v_{n_j}, \bar{\theta} >> ||| \\ = ||| \sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j}) |||$$

Since $\bar{p}_l$ and $\bar{m}_l$ are in $\mathcal{R}_N(x)$, Lemma 3.1 implies that:

$$||| \sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j}) ||| \leq \epsilon \; .$$

Let $\pi : B \to B'$ be the canonical isometric identification of the ball B with the open ball B' in the Euclidean space $\mathbb{R}^d$ centered at 0 with radius $\sqrt{k}/2$. Through this identification, it is clear that for all x in B: $|||x||| = ||\pi(x)||$. Moreover for any pair of points x, x' in $S_{1,\ldots,1}$ such that $x + x'$ is also in $S_{1,\ldots,1}$, we have: $\pi(x + x') = \pi(x) + \pi(x')$. It follows that

$$||| \sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j}) ||| \; = \; || \sum_{j=1}^{l} \pi(c^k_{\bar{\theta}}(v_{n_j})) ||.$$

Finally, since for $1 \leq j \leq l$, $c^k_{\bar{\theta}}(v_{n_j})$ is in $S_{1,\ldots,1}$, we have:

$$\sum_{j=1}^{l} ||\pi(c^k_{\bar{\theta}}(v_{n_j}))|| \; \leq \; 1/\sqrt{k} \cdot || \sum_{j=1}^{l} \pi(c^k_{\bar{\theta}}(v_{n_j})) ||,$$

which implies

$$\sum_{N \leq n, \, n \in I_{1,\ldots,1}, \, \text{even}} l^k_{n_j, \bar{\theta}} \; \leq \; 1/\sqrt{k} \cdot \epsilon \; .$$

This insures that the series $\sum_{n \in I_{1,\ldots,1}, \text{ even}} l^k_{n,\bar{\theta}}$ converges, and consequently the series $\sum_{n \geq 0} l^k_{n,\bar{\theta}}$ converges too.

(ii) Let $(X, \mathcal{A})$ be a linearly recurrent $\mathbb{Z}^d$-action on the Cantor set X. Assume that the combinatorial data is adapted to the action and that the series of $\bar{\theta}$-lengths $\sum_{n \geq 0} l^k_{n,\bar{\theta}}$ converges for some $\bar{\theta}$ in $\mathbb{R}^d$. Fix $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ big enough so that

$$\sum_{n \geq n_0} l^k_{n,\bar{\theta}} < \epsilon.$$

Let us define the map h on the $\mathbb{Z}^d$-orbit of x by,

$$h(\mathcal{A}(\bar{n}, x)) \; = << \bar{n}, \bar{\theta} >>= \; \mathcal{A}^k_{\bar{\theta}}(\bar{n}, 0)$$

for each $\bar{n}$ in $\mathbb{Z}^d$. In order to prove that the map h extends to a continuous map on the closure $\mathbb{O}^k$of the orbit of 0 in $\mathbb{T}^k$, it is enough to prove that h is uniformly continuous, which follows from the continuity of h at x. Consider a point $\bar{p}$ in $\mathcal{R}_{n_0}(x)$ and apply Corollary 2.5. There exists a unique $m_0 \geq n_0$ and a unique sequence $\{\bar{p}_l\}_{0 \leq l \leq m_0 - n_0}$ of points in $\mathbb{Z}^d$ such that:

- m_0 is the smallest $m \geq n_0$ for which $\bar{p} \in \mathcal{P}_{n_0}^m(0)$;
- $\bar{p}_0 = 0$;
- $\bar{p}_l \in \mathcal{P}_{m_0-l}(\bar{p}_{l-1})$ and $\bar{p} \in \mathcal{P}_{n_0}^{m_0-l}(\bar{p}_l),\ \forall\ 1 \leq l \leq m_0 - n_0$;
- $\bar{p}_{m_0-n_0} = \bar{p}$.

Let us write:

$$h(\mathcal{A}(\bar{p}, x)) = \sum_{l=1}^{m_0-n_0} (h(\mathcal{A}(\bar{p}_l, x)) - h(\mathcal{A}(\bar{p}_{l-1}, x))).$$

For any $1 \leq l \leq m_0 - n_0$ both points $\bar{p}_l$ and $\bar{p}_{l-1}$ are in $\mathcal{P}_{m_0-l}(\bar{p}_{l-1})$. Consequently there exists a collection $\{v_{m_0-l,i}\}_{1 \leq i \leq q(m_0-l)}$ of vectors in $\vec{\mathcal{F}}_{m_0-l}(\bar{p}_{l-1})$ such that:

- $q(m_0 - l) \leq k(m_0 - l)$;
- the sequence of points $\{\bar{p}_{l-1,i}\}_{0 \leq i \leq q(m_0-l)}$ defined by:
  - $\bar{p}_{l-1,0} = \bar{p}_{l-1}$;
  - $\bar{p}_{l-1,i} = \bar{p}_{l-1,i-1} + v_{m_0-l,i}$ for $1 \leq i \leq q(m_0 - l)$;
  - $\bar{p}_{l-1,q(m_0-l)} = \bar{p}_l$,

  belongs to $\mathcal{R}_{m_0-l}(x)$.

This yields

$$h(\mathcal{A}(\bar{p}, x)) = \sum_{l=1}^{m_0-n_0} \sum_{i=1}^{p(m_0-l)} (h(\mathcal{A}(\bar{p}_{l-1,i}, x))) - h(\mathcal{A}(\bar{p}_{l-1,i-1}, x))).$$

Now we use the fact that the action is linearly recurrent and that the combinatorial data is adapted to this action. We denote by L a uniform upper bound for the sequence $\{k(n)\}_{n \geq 0}$. We get,

$$|||h(\mathcal{A}(\bar{p}, x))||| \leq L \cdot \sum_{l=1}^{m_0-n_0} l^k_{m_0-l,\bar{\theta}} \leq L \cdot \sum_{n=n_0}^{\infty} l^k_{n,\bar{\theta}} \leq \epsilon.$$

This proves the continuity of h at x. □

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